11th Week Wednesday

3 Mayıs 2023 Çarşamba 14:28

Linear Transformation

$f: A \to A$

$L: V \to V \to$ a linear operator

9. Determine whether the following are linear transformations from P_2 to P_3.

(a) $L(p(x)) = xp(x)$

(b) $L(p(x)) = x^2 + p(x) \to x^2 + ax + b$.

$\to$ (c) $L(p(x)) = p(x) + xp(x) + x^2p'(x)$

$\mathbb{P}_n$: the vector space of all polynomials with degree less than n.

b) $L: \mathbb{P}_2 \longrightarrow \mathbb{P}_3$

$\underbrace{ax+b}_{\in \mathbb{P}_2} \longmapsto \underbrace{x^2 + ax + b}_{\in \mathbb{P}_3}$

1) $L(v_1+v_2) \overset{?}{=} L(v_1) + L(v_2)$

LHS $L(a_1x+b_1 + a_2x+b_2) = L((a_1+a_2)x + (b_1+b_2)) = x^2 + (a_1+a_2)x + (b_1+b_2)$ $\neq$

RHS $L(a_1x+b_1) + L(a_2x+b_2) = x^2 + a_1x + b_1 + x^2 + a_2x + b_2 = 2x^2 + (a_1+a_2)x + b_1+b_2$

NOT a linear transformation.

c) $L: \mathbb{P}_2 \longrightarrow \mathbb{P}_3$

$p(x) \longmapsto p(x) + xp(x) + x^2p'(x) = ax+b + x(ax+b) + x^2 \cdot a$

$ax+b \longmapsto \boxed{2ax^2 + (a+b)x + b}$

Is L a linear transformation?

1) LHS $L(\overset{v_1}{a_1x+b_1} + \overset{v_2}{a_2x+b_2}) = L((a_1+a_2)x + (b_1+b_2)) = 2(a_1+a_2)x^2 + (a_1+a_2+b_1+b_2)x + b_1+b_2$

RHS $L(a_1x+b_1) + L(a_2x+b_2) = 2a_1x^2 + (a_1+b_1)x + b_1 + 2a_2x^2 + (a_2+b_2)x + b_2$

$= (2a_1+2a_2)x^2 + (a_1+b_1+a_2+b_2)x + (b_1+b_2)$ $=\checkmark$

$\forall \alpha \in \mathbb{R}$

2) $L(\alpha(ax+b)) = L((\alpha a)x + (\alpha b)) = 2(\alpha a)x^2 + (\alpha a + \alpha b)x + \alpha b$ $=\checkmark$

$\alpha L(ax+b) = \alpha \cdot (2ax^2 + (a+b)x + b) = \alpha 2a x^2 + \alpha(a+b)x + \alpha b$

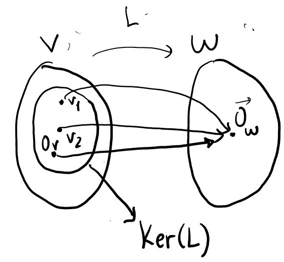
$\Rightarrow L$ is a linear transformation.

Kernel and Range of a Linear Transformation

$L: V \to W$

$\text{Ker}(L)$ $\qquad$ $\text{Range}(L)$, $\text{Im}(L)$, $L(V)$

$L: V \to W$



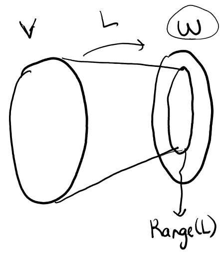
Kernel: $\text{Ker}(L) = \{ v \in V : L(v) = 0_W \}$ → solution set of this system.

a homogeneous system of linear eqns.

! $\text{Ker}(L) \leq V$ $\quad$ ($\text{Ker}(L)$ is a subspace of V).

! $\boxed{0_V \in \text{Ker}(L)}$ $\qquad$! $\boxed{\text{Ker}(L) \neq \emptyset}$

! $L(0_V) = 0_W$



Range: $\text{Range}(L) = \{ \forall w \in W : \exists v \in V,\ L(v) = w \}$

! $\text{Range}(L) \leq W$ $\quad$ ($\text{Range}(L)$ is a subspace of W)

Ex/ $L: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad$ Find a basis for $\text{Ker}(L)$ and $\text{Range}(L)$.

$(x,y) \mapsto (x+y,\ x-y,\ 0)$ $\qquad$ $\dim(\text{Ker}(L)) = ?$ $\quad$ $\dim(\text{Range}(L)) = ?$

$\text{Ker}(L) = \{ v \in \mathbb{R}^2 : L(v) = (0,0,0) \}$

$= \{ (x,y) : L((x,y)) = (0,0,0) \} = \{ (0,0) \}$ → $\boxed{\text{no basis}}$

→ $\dim(\text{Ker}(L)) = 0$

$(x+y,\ x-y,\ 0) = (0,0,0)$

$$\left.\begin{aligned} x+y &= 0 \\ x-y &= 0 \\ 0 &= 0 \end{aligned}\right\} \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$x = 0 \quad y = 0$ $\qquad$ $y = 0 \quad x = 0$

Range: $\text{Range}(L) = \{ (x+y,\ x-y,\ 0) : x, y \in \mathbb{R} \} \leq \mathbb{R}^3$

$$\begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + y \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{v_2}$$

→ $\{v_1, v_2\}$ spans $\text{Range}(L)$

⇒ $\{v_1, v_2\}$ are also lin. indep.

⇒ A basis for $\text{Range}(L)$ $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

$\dim(\text{Range}(L)) = 2$

! For any linear transformation $L: V \to W$;

! For any linear transformation $L : V \to W$;

$$\dim(\operatorname{Ker}(L)) + \dim(\operatorname{Range}(L)) = \dim(V)$$

Ex

$$L : \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x,y) \mapsto (x,x,x)$$

Find a basis for $\operatorname{Ker}(L)$ and $\operatorname{Range}(L)$. Find their dimensions.

$$\operatorname{Ker}(L) : \{ (x,y) : L((x,y)) = (0,0,0) \} = \{ (0,r) : r \in \mathbb{R} \} \leq \mathbb{R}^2$$

$(x,x,x) = (0,0,0)$

$\boxed{x=0}$ ✓

$\boxed{y = r \in \mathbb{R}}$

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$L((0,3)) = (0,0,0)$

$L((0,-2)) = (0,0,0)$

A basis for the kernel $= \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\dim(\operatorname{Ker}(L)) = 1$

$$\operatorname{Range}(L) : \{ (x,x,x) : x \in \mathbb{R} \}$$

$$\begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A basis for the range $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\dim(\operatorname{Range}(L)) = 1$

Ex

$$L : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x,y,z) \mapsto (x+y, y+z)$$

Find a basis for $\operatorname{Ker}(L)$ and $\operatorname{Range}(L)$. Find their dimensions.

Kernel

$$\operatorname{Ker}(L) = \{ (x,y,z) : L((x,y,z)) = (0,0) \}$$

$$(x+y, y+z) = (0,0) \longrightarrow \begin{aligned} x+y &= 0 \\ y+z &= 0 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$z = r \in \mathbb{R}$

$\Rightarrow y = -r$, $x = r$

$$\operatorname{Ker}(L) = \{ (r,-r,r) : r \in \mathbb{R} \}$$

$$\begin{bmatrix} r \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

A basis for $\operatorname{Ker}(L) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\dim(\operatorname{Ker}(L)) = 1$

Range

$$\operatorname{Range}(L) = \{ (x+y, y+z) : x,y,z \in \mathbb{R} \}$$

$$\begin{bmatrix} x+y \\ y+z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \{ v_1, v_2, v_3 \} \text{ spans } \operatorname{Range}(L).$$

^ those lin. indep.? NO!

$$\begin{bmatrix} x+y \\ y+z \end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix} + z\begin{bmatrix}0\\1\end{bmatrix} \rightarrow \{v_1, v_2, v_3\} \text{ spans } Range(L).$$

v_1, $\cancel{v_2}$, v_3

$v_1 + v_3 = v_2$

Are they lin. indep? NO!

A basis for $Range(L) = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$

$\dim(Range(L)) = 2$

Functions

1-1 fnct onto.

$L : V \to W$

1-1 (one-to-one) Lin. Transformation → (Injective)

$\Updownarrow$

$Ker(L) = \{0_V\}$

Onto Linear Transformations → (Surjective)

$Range(L) = W$

$\Updownarrow$

→ $\dim(Range(L)) = \dim(W)$

19. Find the kernel and range of each of the following linear operators on P_3:
(a) $L(p(x)) = xp'(x)$ **(b)** $L(p(x)) = p(x) - p'(x)$
(c) $L(p(x)) = p(0)x + p(1)$

$$L : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$$

$$p(x) \longmapsto p(x) - p'(x) = ax^2 + bx + c \quad -2ax - b$$

$$ax^2 + bx + c \longmapsto ax^2 + (b-2a)x + c - b$$

$Ker(L)$

$$Ker(L) = \{ ax^2 + bx + c \ : \ L(ax^2 + bx + c) = 0x^2 + 0x + 0 \}$$

$$ax^2 + (b-2a)x + (c-b) = 0 \Rightarrow \begin{matrix} a = 0 \\ b - 2a = 0 \\ c - b = 0 \end{matrix} \Rightarrow \begin{matrix} a = 0 \\ b = 0 \\ c = 0 \end{matrix}$$

$$= \{ 0x^2 + 0x + 0 \} \rightarrow \dim = 0, \text{ no basis}$$

$$Range(L) = \{ ax^2 + (b-2a)x + (c-b) \ : \ a, b, c \in \mathbb{R} \}$$

$$ax^2 + (b-2a)x + (c-b) = a(\underbrace{x^2 - 2x}_{v_1}) + b(\underbrace{x - 1}_{v_2}) + c(\underbrace{1}_{v_3})$$

$\{v_1, v_2, v_3\}$

$c_1(x^2-2x)+c_2(x-1)+c_3 \cdot 1 = 0$

$c_1 = 0$

$-2c_1+c_2=0 \quad c_2=0$

$\Rightarrow c_1=c_2=c_3=0 \Rightarrow \{v_1,v_2,v_3\}$ are lin. indep.

A basis for the range $= \{(x^2-2x),(x-1),1\}$

$\dim(\text{Range}(L))=3$

$L(p(x)) = p(0)x + p(1)$

$L: \mathbb{P}_3 \longrightarrow \mathbb{P}_2$

$ax^2+bx+c \longmapsto \boxed{cx+(a+b+c)}$

$\text{Ker}(L) = \{ ax^2+bx+c : \underline{cx+(a+b+c)=0} \}$

$c=0$

$a+b+c=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$b=r\in\mathbb{R}$

$c=0$

$a=-r$

$\text{Ker}(L) = \{-rx^2+rx : r\in\mathbb{R}\}$

$-rx^2+rx = r(x-x^2)$

A basis for the kernel $=\{x-x^2\}$

$\dim(\text{Ker})=1$

$\text{Range}(L) = \{ cx+(a+b+c) : a,b,c\in\mathbb{R}\}$

$cx+(a+b+c) = a(1) + b(\cancel{1}) + c(1+x)$

A basis for Range $= \{1, 1+x\}$ $\quad \dim(\text{Range})=2$